Register No. of the Candidate | | | | | | |

UNIVERSITY OF CALICUT
DEPARTMENT OF MATHEMATICS
M.Phil. Entrance Model Question Paper

Time: 2 hours — Max. Marks: 100

PART - A

Answer **all** questions. Each question carries **2** marks

1. If the statement (P if and only if Q) is not true, then which one of the following is true?
   (a) (Not P) if and only if Q
   (b) P if and only if (Not Q)
   (c) (Not P) if and only if (not Q)
   (d) None of these

2. Which one of the following sets is uncountable?
   (a) The set of all finite subsets of the set of all positive integers
   (b) The set of all subsets, of the set of all positive integers, whose complement is finite
   (c) The set of all infinite subsets of the set of all positive integers
   (d) None of the above

3. Let X be a linearly ordered set containing at least two elements and such that for any two elements x and y such that $x < y$, there exists an element z such that $x < z < y$. Then which one of the following is true?
   (a) X is countable
   (b) X is uncountable
   (c) X is infinite
   (d) None of these

4. Which one of the following is not true for a simple group G with identity 0?
   (a) The only normal subgroups are $\{0\}$ and G
   (b) The only factor groups are $\{0\}$ and G upto isomorphism
   (c) Every non trivial homomorphism from G into any group is one to one.
   (d) The only homomorphism from G into any group is trivial

5. Which one of the following is a finite number?
   (a) Number of algebraic extensions of a finite field
   (b) Number of finite extensions of a finite field
   (c) Number of extensions of an algebraically closed field
   (d) Number of finite extensions of an algebraically closed field

6. If $\mathbb{Q}(\sqrt{\alpha}) = \{a + b\sqrt{\alpha} : a, b \in \mathbb{Q}\}$, for $\alpha = 2, 3$, then which one of the following pairs are not isomorphic structures under the usual operations?
   (a) The fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$
   (b) The groups $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$
   (c) The vector spaces $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ over the field $\mathbb{Q}$
   (d) None of these

7. If G and R denote respectively the group $(\mathbb{Z}, +)$ and the ring $(\mathbb{Z}, +, .)$ and if $Aut(X)$ denotes the group of automorphisms of X, then
   (a) both $Aut(G)$ and $Aut(R)$ are finite
   (b) $Aut(G)$ is finite but $Aut(R)$ is not
   (c) $Aut(R)$ is finite but $Aut(G)$ is not
   (d) both $Aut(G)$ and $Aut(R)$ are infinite

8. If f is a non-zero linear functional on an infinite dimensional vector space V, then the dimension of the nullspace of f is

(a) 0 (b) 1 (c) infinite (d) None of these

9. Let L be the line $\{(x,\ -x):\ x \in \mathbb{R}\}$. Then the vector space $\{a + L:\ a \in \mathbb{R}^2\}$, with respect to the operations $(a_1 + L) + (a_2 + L) = (a_1 + a_2) + L$ and $c(a + L) = ca + L$, for $a_1,\ a_2 \in \mathbb{R}^2,\ c \in \mathbb{R}$, is not isomorphic to

(a) L (b) $\{(x,\ x):\ x \in \mathbb{R}\}$ (c) $\{(x,\ -x):\ y = 2x$ (d) $\mathbb{R}^2$

10. The degree of a minimal polynomial over an n-dimensional vector space over the complex plane $\mathbb{C}$, where $n \geq 1$, may be

(a) 0 (b) n (c) n^2 (d) 2^n

11. If φ denote the Euler's totient Function, define for $x \in \mathbb{R},\ x \geq 0,\ f(x) = \sum_{n \leq x} \varphi(n)\ \ (= 0,\ if\ x < 1)$, where the summation is extended over all positive integers less than or equal to x. Which one of the following is not true?

(a) φ is a monotonic step function
(b) φ is continuous on every interval $[n,\ n+1]$
(c) φ is not continuous on every interval $[n,\ n+1]$
(d) $f(p) = p - 1 + f(p-1)$, for every prime p

12. The eigen spectrum of a bounded linear operator on a normed space is

(a) always bounded and closed
(b) always bounded but need not be closed
(c) always closed but need not be bounded
(d) neither bounded nor closed

13. Which one of the following is not true about the numerical range $\omega(A)$ of a bounded linear operator A on a Hilbert space?

(a) $\omega(A)$ is bounded (b) $\omega(A)$ is convex (c) $\omega(A)$ is closed (d) $\sigma_e(A) \subset \omega(A)$

14. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x^2, y^2 + \cos x)$. Then the derivative of the f at (x, y) is given by:

(a) $\begin{pmatrix} 2x & 0 \\ -\sin x & 2y \end{pmatrix}$ (b) $\begin{pmatrix} 2x & 2y \\ -\sin x & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 2x & 2y \\ -cosx & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 2y & 0 \\ -\sin x & 2y \end{pmatrix}$

15. Let $\Omega \subset \mathbb{R}^3$ be an open set and $f : \Omega \to \mathbb{R}$ be a diffrentiable function such that $(Df)(x) = 0$ for all $x \in \Omega$. Then which of the following is true:

(a) f must be a constant function
(b) f must be constant on connected components of Ω
(c) $f = \chi_\Omega$
(d) The range of the function is a subset of $\mathbb{Z}$.

16. Consider the function $f(x) = \sum_{n=1}^{2019} (\frac{1}{n^5}) \chi_{[0,n/2019]}(x),\ x \in [0,1]$. Then which of the following is not true?

(a) f is Riemann integrable on $[0,1]$
(b) f is Lebesgue integrable on $[0,1]$
(c) f is a continuous function on $[0,1]$
(d) f is a monotone function on $[0,1]$

17. The Cauchy problem $\begin{cases} u_x(x,y) + u_y(x,y) = 0, & (x,y) \in \mathbb{R}^2 \\ u(x,x) = 0, & \text{for all } \ x \in \mathbb{R} \end{cases}$ has

(a) a unique solution
(b) a solution which vanishes at $(2,1)$
(c) Infinitely many solutions
(d) a finite number of solutions

18. The integral equation $y(x) = \lambda \int_0^1 (3x - 2)\xi y(\xi) d\xi$ has

(a) no characteristic number (b) one characteristic number
(c) two characteristic numbers (d) infinitely many characteristic numbers

19. The IVP $y' = 2\sqrt{y}$, $y(0) = a$ has

(a) a unique solution if $a < 0$ (b) no solution if $a > 0$
(c) infinitely many solutions if $a = 0$ (d) unique solution if $a \geq 0$.

20. Which one of the following is false?

(a) There exists a simple connected graph with at least two vertices in which each vertex is a cut vertex
(b) There exists a simple connected graph with at least two vertices in which each edge is a cut edge
(c) There exists a simple connected graph with at least two vertices in which no vertex is a cut vertex and no edge is a cut edge
(d) None of the above

21. Let Σ be a finite nonempty set of symbols. Then which one of the following is true?

(a) Every language on the alphabet Σ is infinite
(b) The set of all languages on the alphabet Σ is countable
(c) The set of all languages on the alphabet Σ is uncountable
(d) None of the above

22. Which of the following is true?

(a) [0, 1] and the unit circle S^1 are homeomorphic (b) [0, 1] and (0, 1) are homeomorphic
(c) [0, 1] and [0, 1) are homeomorphic (d) None of these

23. Which of the following topology on $\mathbb{R}$ is not first countable?

(a) Discrete topology (b) Usual topology
(c) Cofinite topology (d) None of these

24. The value of the integral $\int_C \frac{1}{1+z^2} dz$ where C is the ellipse $x^2 + 4y^2 = 1$

(a) 2π (b) 0 (c) π (d) None of these

25. The number of solutions of the equations $e^z = 1 + 2z$ that satisfy $|z| = 1$ is
(a) 1 (b) 0 (c) ∞ (d) None of these

PART - B

Answer any **ten** questions. Each question carries **5** marks

1. Prove that the set of sequences of positive integers is uncountable.

2. If A is any set, prove that there exists no bijection between A and the power set of A.

3. Give an example of an infinite group which is
(i) not abelian (ii) abelian but not cyclic.
Justify your claim.

4. Does the subring $(\mathbb{Z}, +, .)$ produce a factor ring of $(\mathbb{Q}, +, .)$? Why?

5. Prove or disprove: for any positive integer n, there exists a field extension $F \leq E$, for which the Galois group is isomorphic to $\mathbb{Z}_n$.

6. Let T be the linear operator on $\mathbb{R}^2$ defined by $T(x_1,\ x_2) = (x_1 - x_2,\ 2x_1 + 2x_2)$. Find all subspaces of $\mathbb{R}^2$ which are invariant under T.

7. Define the transpose of a linear transformation from a vector space into a finite dimensional vector space. What is the reason for which it is called the transpose? Justify your claim.

8. Is the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ invertible modulo 13? Why? If yes, find the inverse modulo 13.

9. Prove that no prime $p \equiv 3(mod\ 4)$ can be written as a sum of two squares.

10. Find the fractional linear transformation mapping -1 to 0, ∞ to 1 and i to ∞

11. If $(d_1,\ d_2,\ ...,\ d_n)$ is any sequence of nonnegative integers whose sum is even, show that there exists a graph with $(d_1,\ d_2,\ ...,\ d_n)$ as its degree sequence.

12. Let X be a poset with a smallest element 0 and a largest element 1, such that any nonempty subset of X has a supremum. Prove that X is a complete lattice.

13. Find the general solution for the differential equation $y'' + 4y = 0$. (Your answer should contain a recurrence relation for the power series coefficients.)

14. Find the solution of the Cauchy problem: $3u_x - 4u_y = 0$ with $u(x, x) = x^2 x$.

15. Suppose X is a metric space and $A \subseteq X$ is closed and bounded. Is A necessarily compact? Justify